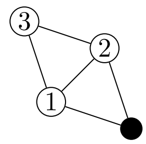
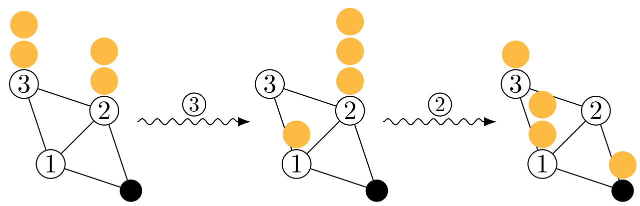
Toppling sandpiles

Travis, Mathcamp 2024

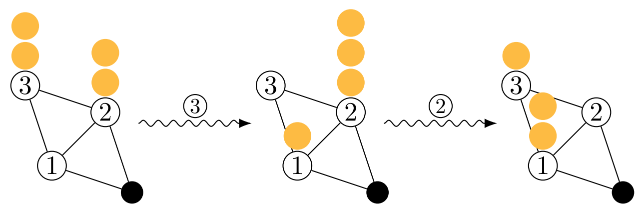
On a strangely mathematical beach nearby, there are piles of sand that behave in very precise way. The piles are arranged like this:



On each pile, there are several grains of sand, but as you know, sandpiles are not very stable. So if there are at least as many grains of sand as edges connected to the pile, the pile of sand will *topple* and send one grain of sand along each edge to the other vertices. This might make other sandpiles topple, and this continues until all the sandpiles are *stable*. For example:

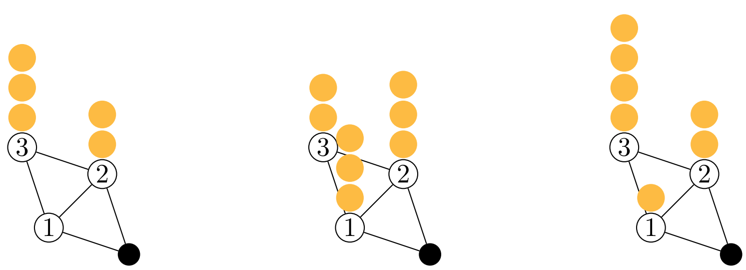


Except the black dot doesn't represent a sandpile, but a *bottomless pit*. Any sand that falls into the pit disappears forever. So the toppling from the previous example actually looks like this:



Since no more sandpiles can topple, that's the final stable configuration.

Question 1. For each of the following examples, what final sandpiles come about after everything topples?



Question 2. Do you notice anything interesting about this process?

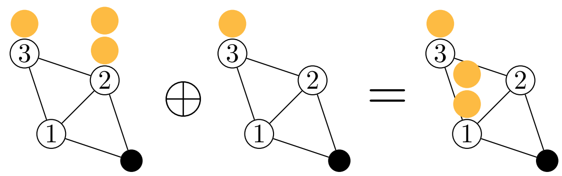
Question 3. Does every sandpile eventually stop toppling?

Question 4. Does it matter in what order the individual piles are toppled?

If a sandpile can't be toppled any more, we call it *stable*. We've found that starting with any sandpile s, there is a *unique* stable sandpile $\overline{s}$ that comes from toppling until there's nothing left to topple. Using this fact, we can define some operations on two sandpiles s_1 and s_2. First, $s_1 + s_2$ is what you get by stacking the sandpiles on top of each other. If s_1 has 2 grains of sand on vertex 1 and s_2 has 3 grains of sand on vertex ①, then $s_1 + s_2$ has 5 grains of sand on vertex ①. And you can get another sandpile by toppling the result:

$$s_1 \oplus s_2 := \overline{s_1 + s_2}.$$

For example:



So $+$ is an operation on sandpiles, while $\oplus$ is an operation on *stable* sandpiles.

Question 5. Is $\oplus$ commutative? associative? Does it have an identity?

Question 6. Does every stable sandpile have an inverse? (If s_1 is a stable sandpile and $s_1 \oplus s_2$ is the identity, then s_2 is an *inverse* of s_1.) If yes, what how can you find it? If no, which elements do have inverses?

Here is a little experiment we can try with sandpiles. Start with a flat, empty sandpile, and add a grain of sand at a random vertex. Then add another grain of sand to a random vertex and topple until you get to a stable sandpile. Then add another random grain and topple; then again and again and again. If you run this experiment and keep track of which stable sandpiles occur, you'll get results like this:

Sandpile	**Trials**									
(0, 0, 0)	0	0	0	0	0	0	0	0	0	0
(0, 0, 1)	0	0	0	1	0	1	0	1	1	0
(0, 1, 0)	0	0	1	0	0	0	1	0	0	1
(0, 1, 1)	0	0	0	0	0	0	0	0	0	1
(0, 2, 0)	0	0	1	0	0	0	1	0	0	0
(0, 2, 1)	11	8	11	11	16	14	13	12	9	16
(1, 0, 0)	1	1	0	0	1	0	0	0	0	0
(1, 0, 1)	0	1	0	0	1	0	0	1	0	0
(1, 1, 0)	0	0	0	1	0	1	0	0	1	0
(1, 1, 1)	0	0	0	0	1	0	0	0	0	0
(1, 2, 0)	12	14	13	15	9	11	10	12	18	15
(1, 2, 1)	16	14	16	12	13	7	13	12	12	13
(2, 0, 0)	1	0	0	0	0	0	0	0	0	0
(2, 0, 1)	15	11	9	16	8	17	7	10	10	15
(2, 1, 0)	7	12	15	13	11	16	16	17	9	11
(2, 1, 1)	17	15	13	10	7	15	14	15	6	8
(2, 2, 0)	6	11	9	12	16	12	12	10	21	10
(2, 2, 1)	14	13	12	9	17	6	13	10	13	10

Each column represents a single trial, and each entry is the number of times that a specific stable sandpile appeared in the trial. (The notation (a, b, c) means that there are a grains of sand on vertex ①, b grains of sand on vertex ②, and c grains of sand on vertex ③.)

Question 7. What do you notice about this table?

The set of stable sandpiles with the operation $\oplus$ isn't a group. But it turns out that these "recurring" sandpiles *are* a group! And that's strange—what could the identity even be???

We'll find out tomorrow :)

Bonus

Question 8. What happens if we replace the diamond graph with a different one? Do your answers to Questions 3–6 change?

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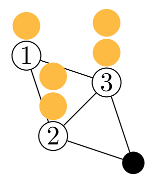
We use the notation $\overline{s}$ to denote the unique sandpile you get from toppling s.

Exercise 9. Prove that $\overline{s_1 + s_2} = \overline{\overline{s_1} + \overline{s_2}}$.

A stable sandpile s is called *recurrent* if for every sandpile a there is another sandpile b such that $s = a \oplus b$.

Exercise 10. Show that this sandpile is recurrent:



We call this sandpile $s_{\max}$.

Exercise 11. Show that s is recurrent if and only if there is a sandpile b such that $s = s_{\max} \oplus b$.

Question 12. Remember the experiment from yesterday, where we repeatedly dropped a random grain of sand and stabilized. Why do you expect to eventually reach a recurrent sandpile? Explain why, once you reach a recurrent sandpile, every sandpile afterward is also recurrent.

Let $\mathcal{S}$ denote the set of recurrent sandpiles.

Theorem. *$\mathcal{S}$ with the operation $\oplus$ is a group.*

This is a bold claim: The "obvious" identity element—with 0 grains at every vertex—is not a recurrent sandpile! So let's see what's really going on.

We know that $\oplus$ is commutative and associative, so we need to prove that there is an identity and that every recurrent sandpile has an inverse.

Exercise 13. Prove that $e := \overline{2s_{\max} - \overline{2s_{\max}}}$ is a recurrent sandpile and is the identity. (What is this sandpile explicitly?)

Exercise 14. Show that every recurrent sandpile has an inverse: If s is recurrent, then there is another recurrent sandpile a so that $s \oplus a = e$.

Question 15. Suppose we change the graph. How should we define $s_{\max}$ so that Exercise 11 remains true? What happens with Exercises 13 and 14?

Bonus questions

Exercise 16. Prove that s is recurrent if and only if $e \oplus s = s$.

Exercise 17. Suppose that s is recurrent. Show that on any pair of adjacent vertices (in which neither is the pit), at least one of those vertices has at least one grain of sand.

Exercise 18 (Challenge). Suppose that T is a set and $*$ is an operation on T. The pair $(T, *)$ is a *commutative monoid* if $*$ is associative and commutative and has an identity element. The "sum" of two subsets $A, B \subseteq T$ is defined as

$$A * B := \{a * b : a \in A \text{ and } b \in B\}.$$

A subset $I \subseteq T$ is called an *ideal* if $I * T = I$. An ideal is *minimal* if it does not contain any smaller ideal. Suppose that T is a finite commutative monoid.

(i) Prove that the intersection of two ideals is an ideal.
(ii) Prove that T has exactly one minimal ideal.

(iii) Let M denote the minimal ideal. Prove that $c \in M$ *if and only if* for every $a \in T$, there is a $b \in T$ such that $a * b = c$. Moreover, prove that if $c \in M$, then for any $a \in T$, there is a $b \in M$ such that $c = a * b$.

(iv) Prove that there is an element $e \in M$ such that $e^2 = e$. [Hint: consider the sequence $a, a^2, a^3, \ldots$ for some element $a \in M$.]

(v) Prove that $e * a = a$ for any element $a \in M$.

(vi) Prove that $(M, *)$ is an abelian group.

(vii) Suppose that T is the set of stable sandpiles. Prove that

1. $M = \mathcal{S}$, the set of recurrent sandpiles.
2. M is a *principal ideal*, meaning that there is an element $c \in T$ such that $M = \{c\} * T$.

fun facts

Fun fact #1! The number of elements of the sandpile group is equal to the number of spanning trees of the graph.

WHAT??? WHAT??? WHAT??? WHAT??? WHAT??? WHAT??? WHAT??? WHAT??? WHAT??? WHAT??? WHAT???

It's true.

Fun fact #2! If you change the location of the pit, the sandpile groups remain isomorphic.

WHAT??? WHAT??? WHAT??? WHAT??? WHAT??? WHAT??? WHAT??? WHAT??? WHAT??? WHAT??? WHAT???

Also true.