The Ra(n)do(m) Graph

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1. random graphs

The standard way to produce a random graph is

Definition 1. For each $n \in \mathbb{N}$ and $0 < p < 1$, the *Erdős-Rényi random graph* $G(n,p)$ is generated by taking n vertices and adding each edge independently with probability p.

For example, to generate $G(n, 1/2)$, start with n vertices. Then, for each pair of vertices, flip a coin. If it comes up heads, draw an edge; if tails, do nothing. This way of generating a random graph turns out to be useful in nonconstructive combinatorial proofs; Yuval probably talked about this in his Probabilistic Method class. And there are interesting questions that you can ask about the model itself. For example, how big does p have to be for it to be more likely than not that $G(n,p)$ is connected?

But we're not going to worry about any of that. We'll skip right to the infinite stuff:

Definition 2. For each $0 < p < 1$, the *countable Erdős-Rényi random graph* $G(\mathbb{N}, p)$ is generated by taking infinitely many vertices, one for each natural number, and adding an edge between each pair of vertices independently with probability p.

The first thing that we can note is this:

Proposition 3. *$G(\mathbb{N}, p)$ contains any given finite graph H as an induced subgraph with probability 1.*

Proof. Say that H has n vertices and e edges. The probability that H is an induced subgraph on the vertex set $\{1, 2, \ldots, n\}$ is $p^e(1-p)^{\binom{n}{2}-e}$. The same for the vertex set $\{n+1, n+2, \ldots, 2n\}$. So the probability that H is *not* an induced subgraph of $G(\mathbb{N}, p)$ is at most

$$\prod_{i=1}^{\infty} \left(1 - p^e(1-p)^{\binom{n}{2}-e}\right) = 0. \qquad \square$$

2. alice's restaurant

The crux of countable random graphs is this somewhat odd property.

Definition 4. A graph has *Alice's restaurant property* if, for every collection of vertices $v_1, \ldots, v_n$ and $u_1, \ldots, u_m$, there is a vertex x that is adjacent to every vertex $v_1, \ldots, v_n$ and is not adjacent to any vertex $u_1, \ldots, u_m$. We call the vertex x *Alice-adjacent* to this vertex selection.

Why Alice's restaurant? Well, in 1967, the folk singer Arlo Guthrie (son of Woody Guthrie, he himself of *This Land is Your Land* fame) released a song called *Alice's Restaurant*, whose second stanza starts with the line "You can get anything you want at Alice's Restaurant."[1] Then some mathematician thought of this lyric and decided that was enough motivation to bequeath the name.

Proposition 5. *With probability 1, the graph $G(\mathbb{N}, p)$ has Alice's restaurant property.*

Proof. The probability that the vertex z in $G(\mathbb{N}, p)$ is connected to each vertex $v_1, \ldots, v_n$ and to none of the vertices $u_1, \ldots, u_m$ is $p^n(1-p)^m$. So the probability that there is *no* vertex that is Alice-adjacent to these vertices is at most

$$\prod_{z} \left(1 - p^n(1-p)^m\right) = 0.$$

[1] Incidentally, the first stanza of the song is "This song is called 'Alice's Restaurant.' It's about Alice, and the Restaurant, but 'Alice's Restaurant' is not the name of the restaurant, that's just the name of the song. That's why I call the song 'Alice's Restaurant.' "

Let $A(v_1, \ldots, v_n; u_1, \ldots, u_m)$ denote the property that there is a vertex adjacent to all of $v_1, \ldots, v_n$ and none of $u_1, \ldots, u_m$. The probability that $G(\mathbb{N}, p)$ does not have Alice's restaurant property is at most

$$\sum_{\substack{u_1,\ldots,u_m \\ v_1,\ldots,v_n \in \mathbb{N} \\ m,n \in \mathbb{N}}} A(v_1, \ldots, v_n; u_1, \ldots, u_m) = 0,$$

so $G(\mathbb{N}, p)$ has Alice's restaurant property with probability 1. □

In fact, Alice's restaurant property contains the seeds of its own strengthening.

DEFINITION 6. A graph satisfies *Alice's industrial restaurant property* (or AIRP) if, for every $v_1, \ldots, v_n$ and $u_1, \ldots, u_m$, there are infinitely many vertices z that are adjacent to $v_1, \ldots, v_n$ and not adjacent to $u_1, \ldots, u_m$.

LEMMA 7. *Any countable graph that satisfies Alice's restaurant property also satisfies Alice's industrial restaurant property.*

Proof. Suppose that $z_1, \ldots, z_k$ are vertices adjacent to $v_1, \ldots, v_n$ and not adjacent to $u_1, \ldots, u_m$. Then Alice's restaurant property implies that there is a vertex z_{k+1} that is adjacent to $v_1, \ldots, v_n, z_1, \ldots, z_k$ and not adjacent to $u_1, \ldots, u_m$. So we can build an infinite list $z_1, z_2, \ldots$ of vertices that satisfy Alice's restaurant property. □

Now, we get to the punchline:

THEOREM 8. *Any two countable graphs with Alice's restaurant property are isomorphic.*

Proof. We use the "back-and-forth method," which is, in this case, really just a way of saying "just do it." Suppose that G and H are both countable graphs with Alice's restaurant property.

We define a map $f : V(G) \to V(H)$ inductively as follows. Begin with $A_0 = B_0 = \emptyset$. At step n,

- Let a_{k_n} be the vertex of $V(G) \setminus A_{n-1}$ with least index. Define $S_n = \{f(a_i) : a_i \in A_{n-1} \text{ and } (a_i, a_{k_n}) \in E(G)\}$ and $T_n = \{f(a_i) : a_i \in A_{n-1} \text{ and } (a_i, a_{k_n}) \notin E(G)\}$. Because H has Alice's industrial restaurant property, there is a vertex $b_{r_n} \in V(H) \setminus B_{n-1}$ with minimal index so that b_{r_n} extends (S_n, T_n) (since B_{n-1} is finite). We set $f(a_{k_n}) = b_{r_n}$. Define $A'_n = A_{n-1} \cup \{a_{k_n}\}$ and $B'_n = B_{n-1} \cup \{b_{r_n}\}$.
- Let $b_{r'_n}$ be the vertex in $V(H) \setminus B'_n$ with least index. Define $S'_n = \{a_i \in A'_n : (f(a_i), b_{r'_n}) \in E(H)\}$ and $T'_n = \{a_i \in A'_n : (f(a_i), b_{r'_n}) \notin E(G)\}$. Since G has Alice's industrial restaurant property, there is a vertex $a_{k'_n} \in V(H) \setminus A'_n$ with minimal index so that $a_{k'_n}$ extends (S'_n, T'_n). We set $f(a_{k'_n}) = b_{r'_n}$ and then define $A_n = A'_n \cup \{a_{k'_n}\}$ and $B_n = B'_n \cup \{b_{r'_n}\}$.

By the choice of the vertices, at the end of each step, f is an isomorphism between $G[A_n]$ and $H[B_n]$. Moreover, $\{1, 2, \ldots, n\} \subseteq A_n, B_n$, so the inductively defined function f is a bijection from $V(G)$ onto $V(H)$. Thus f is an isomorphsim between G and H. □

In other words, if you create two countable Erdős-Rényi random graphs, with probability 1 they will be isomorphic.

This is astounding—you should be gasping with amazement right now. Think about how this compares to finite graphs. The probability that two random graphs $G(n, p)$ are isomorphic is small, and this probability *tends to 0* as n goes to infinity. And yet, there's only one countably infinite random graph. Stupefied yet?

DEFINITION 9. The unique countable graph with Alice's restaurant property is called the *Rado graph*. We'll denote the Rado graph by R.

3. ROBUSTNESS

A *switch* consists of deleting or adding an edge.

PROPOSITION 10. *Any graph obtained from R by a finite number of switches is isomorphic to R.*

Proof. We only need to show that the new graph R' also satisfies Alice's restaurant property; if it does, then Theorem 8 shows that R' and R are isomorphic. So fix $v_1, \ldots, v_n$ and $u_1, \ldots, u_m$. We know that R satisfies AIRP, and there are only finitely many switches. So among all vertices satisfying Alice's restaurant property in R, there is a vertex z such that the edge switches relating R and R' do not change any adjacencies (or non-adjacencies) between z and the u_i and v_i. This means that z also satisfies Alice's restaurant property in R'. □

Here are two more in the same style:

Proposition 11. *Any graph obtained from R by deleting a finite number of vertices is isomorphic to R.*

Proposition 12. *The graph obtained from R by switching* all *edges at a single vertex is isomorphic to R.*

And one more of a slightly different flavor:

Proposition 13. *If the vertices of R are partitioned into finitely many sets, the induced graph on one part of the partition is isomorphic to R.*

See problem 3 for a proof; problem 4 asks you to explain why it's not true that *all* induced graphs are isomorphic to R.

In fact, this property is (basically) unique to the Rado graph:

Proposition 14. *If G is a countable graph that has the property that, whenever the vertex set is partitioned into two parts, there is one parts whose induced subgraph is isomorphic to G, then G is either the empty graph, the complete graph, or R.*

Proof. Suppose that G is not empty. Let X be the set of isolated vertices in G and $Y = V(G) \setminus X$. Since G is not empty, $G \ncong X$; but if $X \neq \emptyset$, then $G \ncong Y$, either, since Y has no isolated vertices. We conclude that if G is not empty, it has no isolated vertices. Similarly, if G is not complete, it has no vertex connected to every other vertex.

We now assume that G is neither complete nor empty. Let k-ARP denote the statement that, for every $v_1, \ldots, v_n$ and $u_1, \ldots, u_m$ with $n + m \leq k$, there is a vertex z connected to every $v_1, \ldots, v_n$ and to none of $u_1, \ldots, u_m$. We will prove that G has Alice's restaurant property (and is therefore isomorphic to R) by proving that G satisfies k-ARP for every $k \in \mathbb{N}$.

The first paragraph of the proof implies that G has 1-ARP. Suppose that G has k-ARP for some $k \geq 1$ and let $V = \{v_1, \ldots, v_n\}$ and $U = \{u_1, \ldots, u_m\}$ be some distinct vertices of G with $m + n = k + 1$. It's possible that $m = 0$ or $n = 0$, but since $k \geq 1$, we can choose two nonempty sets A and B so that $A \sqcup B = U \cup V$. Call a vertex $x \in V(G)$ *bad for $u \in U$* if it is connected to u, and call it *bad for $v \in V$* if it is *not* connected to v. Define

$$X = \{z \in V(G) \setminus (A \cup B) : z \text{ is bad for some vertex in } A\} \cup A$$
$$Y = \{z \in V(G) \setminus (A \cup B) : z \text{ is bad for some vertex in } B\} \cup B.$$

If $X \cup Y = V(G)$, then G can be partitioned into X and $Y \setminus X$. By assumption, either $G \cong X$ or $G \cong Y \setminus X$; you can check that this violates k-ARP. Otherwise, there is a vertex $z \in V(G) \setminus (X \cup Y)$, which means that z is connected to every vertex in V and no vertex in U. So G has $(k+1)$-ARP. □

4. The 0–1 law for random graphs

Let P be a property of graphs. (That is, every graph either does or does not have property P, but not both.) Let $\pi_n(P)$ denote the probability that $G(n,p)$ has property P. If $\lim_{n\to\infty} \pi_n(P)$ exists, we denote the limit by $\pi(P)$.

A *first-order statement* in the language of graph theory is a (finite) statement that uses only

- quantifies $\exists$ (there exists) and $\forall$ (for all);
- variables $x_1, x_2, \ldots$ (for vertices)
- relations $=$ (equality) and $\sim$ (adjacency)
- usual Boolean connectives $\wedge$ (and), $\vee$ (or), $\neg$ (negation), and $\Rightarrow$ (implies)

For example, the statement that a graph has no isolated vertices is:

$$\forall x \exists y \ (x \sim y).$$

The statement that a graph contains a triangle is

$$\exists x, y, z \ \big((x \sim y) \wedge (y \sim z) \wedge (z \sim x)\big).$$

And the statement that the diameter of a graph is at most 2 is

$$\forall x, y \exists z \ \Big((x = y) \vee (x \sim y) \vee \big((x \sim z) \wedge (z \sim y)\big)\Big).$$

There is a very surprising theorem about first-order statements and graphs:

THEOREM 15 (0–1 Law). *If P is a first-order statement in the language of graph theory, then $\lim_{n\to\infty} \pi_n(P)$ exists and either $\pi(P) = 0$ or $\pi(P) = 1$.*

In other words, every first-order statement about graphs either is true for almost every graph or is false for almost every graph.

Let's prove it.

For each $m, n \in \mathbb{N}$, let $\sigma_{m,n}$ denote the first-order statement

$$\forall x_1, \ldots, x_m \forall y_1, \ldots, y_n \left(\bigvee_{\substack{1\le i\le m \\ 1\le j\le n}} (x_i = y_j) \vee \Big(\exists z \bigwedge_{1\le i\le m} (z \sim x_i) \wedge \bigwedge_{1\le j\le n} (z \not\sim y_i)\Big) \right).$$

So $\sigma_{m,n}$ is the Alice's restaurant property with m vertices in one group and n vertices in the other. This notation can be extended to when $m = 0$ or $n = 0$; then $\sigma_{m,0}$ means that every set of m vertices have a common neighbor, and $\sigma_{0,n}$ means that for every set of n vertices there is a vertex not connected to any of them.

Now, consider the collection of statements $\sigma_{m,n}$ for $m, n \ge 0$. We can take these as the axioms of a theory, which we'll call $\mathcal{T}$. This theory is *complete*: For every first-order statement S, either S or $\neg S$ is provable from the axioms $\sigma_{m,n}$.[2]

Proof sketch of Theorem 15. Let S be a first-order statement in the language of graph theory. If S is provable in $\mathcal{T}$, then there is a proof of S using only the axioms $\sigma_{m,n}$. Since proof is finite, this proof uses only a finite list of the $\sigma_{m,n}$; call them $\sigma^1, \ldots, \sigma^k$.

What does this mean? Every finite graph that satisfies $\bigwedge_{1\le i\le k} \sigma^i$ will satisfy property S. Applying the contrapositive, any graph that satisfies property $\neg S$ will satisfy $\bigvee_{1\le i\le k} \neg\sigma^i$. Then

$$\mathbb{P}\big(G(n,p) \text{ does not satisfy } S\big) \le \sum_{1\le i\le k} \mathbb{P}\big(G(n,p) \text{ does not satisfy } \sigma^i\big)$$

We have

$$\mathbb{P}\big(G(n,p) \text{ does not satisfy } \sigma_{r,s}\big) \le \binom{n}{r}\binom{n-r}{s}(1 - p^{-r-s})^{n-r-s},$$

which tends to 0 as $n \to \infty$. So

$$\mathbb{P}\big(G(n,p) \text{ does not satisfy } S\big) \longrightarrow 0.$$

So $\pi_n(S) \to 1$.

If S is not provable in $\mathcal{T}$, then $\neg S$ is provable (since $\mathcal{T}$ is complete), so $\pi_n(\neg S) \to 1$, meaning that $\pi_n(S) \to 0$. □

This is a very impressive result. It's important, however, to note what it *does not* say. There is no first-order statement, for example, that says a graph has an *even* number of triangles. There is no first-order statement, in fact, that says the graph is connected! One must be careful to not apply this theorem too far.

5. PROBLEMS

1. Prove that the complement of R (replacing each edge by a non-edge and vice versa) is isomorphic to R.

[2] Why? This is a consequence of (1) the fact that there is only one countable graph that satisfies all the axioms and (2) Gödel's Completeness Theorem. For more details, see chapters 0 and 1 of Joel Spencer's *The Strange Logic of Random Graphs*.

2. Prove Propositions 11 and 12.
3. Prove Proposition 13. [HINT: Use Theorem 8 and contradiction.]
4. Find a countable subset of vertices of R whose induced subgraph is not isomorphic to R. Use this to show that R can be partitioned into two sets where only one induced subgraph is isomorphic to R.
5. Let S be a set of positive integers. The *S-circulant graph* on the vertex set $\mathbb{N}$ has edges between i and $i + s$ for every $s \in S$ (and no other edges). Find a set S for which the S-circulant graph is isomorphic to the Rado graph.
6. Construct a graph on the vertex set $\mathbb{N}$ by connecting vertices i and j (with $i < j$) if and only if the ith bit in the binary expansion of j is nonzero. (For example, 0 is connected to all odd numbers, and 1 is connected to any number congruent to 2 or 3 modulo 4.) Show that this graph is isomorphic to the Rado graph.
7. A *linear order* on a set X is a relation $\preceq$ such that
   - $x \preceq x$ for all $x \in X$;
   - if $x \preceq y$ and $y \preceq x$, then $x = y$;
   - for any $x, y \in X$, either $x \preceq y$ or $y \preceq x$ (or both); and
   - if $x \preceq y$ and $y \preceq z$, then $x \preceq z$.

   A linear order is *dense* if for any $x, y \in X$ with $x \preceq y$, there is an element $z \in X$ such that $x \preceq z \preceq y$. A point $x \in X$ is an *maximum* if $y \preceq x$ for every $y \in X$, and it is a *minimum* if $x \preceq y$ for every $y \in X$; it is an *endpoint* if it is either a maximum or a minimum.

   Suppose that $(X, \preceq_1)$ and $(Y, \preceq_2)$ are two dense linear orders without endpoints and that both X and Y are countable sets. (For example, $\mathbb{Q}$ is a dense linear order without endpoints.) Prove that these linear orders are isomorphic (that is, there is a bijection $\varphi\colon X \to Y$ such that, for every $x, y \in X$, we have $x \preceq_1 y$ if and only if $\varphi(x) \preceq_2 \varphi(y)$).[3]

[3] What does this have to do with countable random graphs? Solve the problem to find out!