Graph inequalities by magic

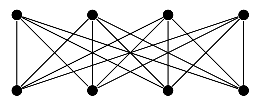
Travis, Mathcamp 2024

Suppose I have three graphs: F_1, F_2, and G. What does knowing something about the number of times that F_1 appears in G tell you about the number of times F_2 appears in G? This is the central question of this class.

Let's start out with a specific example.

> **Activity:** ($F_1 = K_2$, $F_2 = K_3$) Find a graph with n vertices and as many edges as possible, but no triangles. (Start with $n = 4$.)

As you may have found, it seems that a good way to make a graph with lots of edges but no triangles is to divide the vertices into two groups and only draw the edges *between* the two groups. This is an *bipartite graph* $K_{a,b}$:



The bipartite graph $K_{a,b}$ has ab edges. If $a + b = n$, then ab is maximized when $a = b = n/2$ (as long as n is even), so we have found a graph with no triangles and $n^2/4$ edges. In fact, this graph has the most edges possible:

Theorem 1 (Mantel's Theorem). *If a graph with n vertices does not contain a triangle, then it has at most $n^2/4$ edges.*

You'll see a proof of Mantel's theorem in the problems (and a different proof in tomorrow's problems!). Our next goal is to move to a more general framework.

Definition 2. Suppose that F and G are two graphs. A map $\varphi\colon V(F) \to V(G)$ is a *graph homomorphism* if $\varphi(x)\varphi(y)$ is an edge in G whenever xy is an edge in F. The number of graph homomorphisms from F to G is denoted by $\mathsf{hom}(F, G)$.

A graph homomorphism is just a way of "locating" a copy of F inside G.

> **Activity:** Calculate
> - $\mathsf{hom}(K_2, K_3)$
> - $\mathsf{hom}(K_3, K_4)$
> - $\mathsf{hom}(C_4, C_4)$
> - $\mathsf{hom}(K_r, K_n)$

> **Activity:** Prove that, for every graph G,
> - If F_1 and F_2 have the same set of vertices, but $F_1 \subseteq F_2$, then $\mathsf{hom}(F_1, G) \geq \mathsf{hom}(F_2, G)$.
> - $\mathsf{hom}(F_1 \sqcup F_2, G) = \mathsf{hom}(F_1, G)\mathsf{hom}(F_2, G)$, where $F_1 \sqcup F_2$ is the *disjoint union* of F_1 and F_2.

Translated into homomorphism numbers, Mantel's theorem says that if $\mathsf{hom}(K_3, G) = 0$, then $\mathsf{hom}(K_2, G) \leq \frac{1}{4}n^2$. A central question of *extremal graph theory* is trying to determine relationships like this.

We can flip the relationship around: Perhaps we know the number of edges of G and want to find a bound on the number of triangles G contains. In other words, can we find an upper bound for $\mathsf{hom}(K_3, G)$ in terms of $\mathsf{hom}(K_2, G)$?

> **Activity:** Suppose that G has 10 edges; what is the maximum number of triangles it can have? What if G has m edges?

In the activity, it seems that the best thing you can do is push all the edges together, to make something close to a complete graph. How good is this strategy in general? Well, $\mathsf{hom}(K_2, K_n) = n(n-1)$ and $\mathsf{hom}(K_3, K_n) = n(n-1)(n-2)$. So $\mathsf{hom}(K_3, K_n) \approx \mathsf{hom}(K_2, K_n)^{3/2}$. In fact, this *is* the best possible:

Theorem 3 (Kruskal-Katona). $\mathsf{hom}(K_3, G) \le \mathsf{hom}(K_2, G)^{3/2}$.

To prove the theorem, we'll need a very useful inequality:

Lemma 4 (Cauchy–Schwarz). *If $a_1, \dots, a_k$ and $b_1, \dots, b_k$ are real numbers, then*

$$\sum_{i=1}^{k} a_i b_i \le \sqrt{\sum_{i=1}^{k} a_i^2}\,\sqrt{\sum_{i=1}^{k} b_i^2}.$$

The (optional) homework problems include two proofs of this inequality. For now, on to the juicy stuff!

Proof of Kruskal–Katona. Given two vertices x, y, we define

$$a(x,y) = \begin{cases} 1 & \text{if there is an edge between } x \text{ and } y \\ 0 & \text{if not.} \end{cases}$$

Three vertices form a triangle in G if and only if $a(x,y)a(y,z)a(z,x) = 1$. Therefore, one way to write $\mathsf{hom}(K_3, G)$ is like this:

$$\mathsf{hom}(K_3, G) = \sum_{x,y,z \in V(G)} a(x,y)a(y,z)a(z,x) = \sum_{y,z \in V(G)} a(y,z)\Big(\sum_{x \in V(G)} a(x,y)a(z,x)\Big).$$

Applying Cauchy–Schwarz to the last expression, where $a_i = a(y,z)$ and $b_i = \sum_x a(x,y)a(z,x)$, provides an upper bound of

$$\sqrt{\sum_{y,z \in V(G)} a(y,z)^2}\,\sqrt{\sum_{y,z \in V(G)} \Big(\sum_{x \in V(G)} a(x,y)a(z,x)\Big)^2}.$$

Now we look at each sum under the square root. Since $a(y,z)$ is either 1 or 0, the left sum is equal to

$$\sum_{y,z \in V(G)} a(y,z),$$

which is simply the number of edges in G, or $\mathsf{hom}(K_2, G)$. If we expand the square in the right square root, we get

$$\sum_{y,z \in V(G)} \Big(\sum_{x \in V(G)} a(x,y)a(z,x)\Big)\Big(\sum_{w \in V(G)} a(w,y)a(z,w)\Big) = \sum_{x,y,z,w} a(x,y)a(y,w)a(w,z)a(z,x).$$

This is the number of copies of C_4 in G. Since $2K_2 \subseteq C_4$, we have

$$\mathsf{hom}(C_4, G) \le \mathsf{hom}(2K_2, G) = \mathsf{hom}(K_2, G)^2.$$

Now we can put everything together:

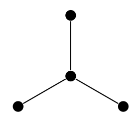
$$\mathsf{hom}(K_3, G) \le \sqrt{\mathsf{hom}(K_2, G)}\sqrt{\mathsf{hom}(C_4, G)} \le \sqrt{\mathsf{hom}(K_2, G)}\sqrt{\mathsf{hom}(K_2, G)^2} = \mathsf{hom}(K_2, G)^{3/2}.$$ □

You might look at this proof and say

but tomorrow, we'll translate this into a one-line proof by picture that seems easy-peasy.

PROBLEMS

1. Here are some interpretations of homomorphism numbers, to get even more used to them! (Think about these problems first.)
   (a) Prove that $\mathsf{hom}(P_k, G)$ is the number of walks of length k in G. (P_k is the *path* with k vertices; a *walk* is a sequence of vertices that are connected by an edge.)
   (b) If you paint each vertex of a graph one color, using a palette of r colors, what you get is called an *r-coloring* of the graph. (You don't have to use all r colors.) If every pair of vertices that are connected by an edge have different colors, the r-coloring is called *proper*. Prove that $\mathsf{hom}(G, K_r)$ is the number of proper r-colorings of G.
   (c) The *complement* of G is a new graph $\overline{G}$ with the same set of vertices. There is an edge between two vertices in $\overline{G}$ if and only if there is *not* an edge between those two vertices in G. For example, these two graphs are complements:  and . A set of vertices is called *independent* if there is no edge between any pair of vertices in the set. Prove that $\mathsf{hom}(K_r, \overline{G})$ is the number of (ordered) independent sets of size r in G.
   (d) The *star graph* S_k has a single vertex connected to k other vertices. The graph S_3 looks like this:



   Prove that
$$\mathsf{hom}(S_k, G) = \sum_{x \in V(G)} \deg(x)^k.$$

The rest of these problems are just for fun. They're related to the material in the class, but they won't be used in class tomorrow. Think about any that you find interesting!

2. Let's prove Mantel's theorem!
   (a) Suppose that G is a graph with n vertices that contains no triangle, and let x, y be two vertices in G that are connected by an edge. Prove that there are at most $n-1$ edges that connect to either x or y.
   (b) Prove Mantel's theorem by induction.
   (c) (Challenge) There is a strengthening of Mantel's theorem, called *Turán's theorem*, which says that any graph with n vertices that does not contain a copy of K_{r+1} has at most $(1 - \frac{1}{r})\frac{n^2}{2}$ edges. Prove Turán's theorem by induction.
3. If you've never seen a proof of the Cauchy–Schwarz inequality and would *like* to, here is an inductive proof.
   (a) Prove the inequality for $k = 2$, that is: $(a_1b_1 + a_2b_2)^2 \le (a_1^2 + a_2^2) + (b_1^2 + b_2^2)$.
   (b) Assume the inequality is true for some k, and prove that the inequality is true for $k+1$. [Hint: How can you apply the inductive hypothesis to the sum $\sum_{i=1}^k a_i b_i$?]
4. Here is another proof of Cauchy–Schwarz, this time without induction!
   (a) Prove that $\sum_{i=1}^k a_i b_i \le \frac{1}{2}\sum_{i=1}^k a_i^2 + \frac{1}{2}\sum_{i=1}^k b_i^2$.
   (b) Suppose we know that $\sum_{i=1}^k a_i^2 = 1$ and $\sum_{i=1}^k b_i^2 = 1$. Prove the Cauchy–Schwarz inequality in this case.
   (c) Now prove the Cauchy–Schwarz inequality without the restriction in part (b).
5. (Challenge; not related to class except it's a fun problem about homomorphisms.) Let G be a directed graph, $\overrightarrow{P}_n$ denote the directed path on n nodes, and $\overrightarrow{K}_n$ denote the transitive tournament on $[n] := \{1, 2, \ldots, n\}$ where $(i, j) \in E(\overrightarrow{K}_n)$ if and only if $i < j$. Prove that G has a homomorphism into $\overrightarrow{K}_n$ if and only if $\overrightarrow{P}_{n+1}$ has no homomorphism into G.

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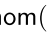
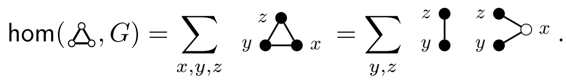
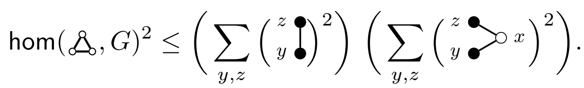
Let's go over that proof of Kruskal–Katona again. Remember, we're trying to prove that hom(, $G)^2 \leq$ hom(, $G)^3$ for every graph G. One way we can rewrite the first step is

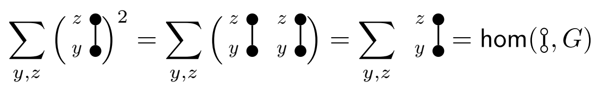


In the last sum, we think of "gluing together" the filled-in vertices that appear twice. The open vertex indicates that we're hiding a sum over x.

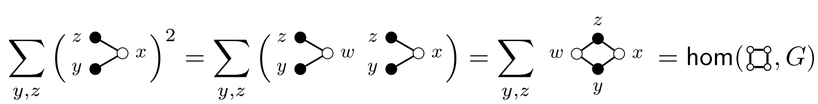
When we take the square and apply Cauchy–Schwarz, we get



Now, look at both of the sums. For the first,

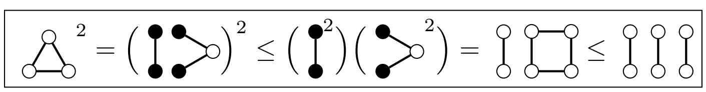


by "gluing together" the corresponding vertices. For the second,

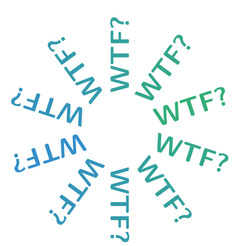


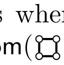
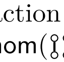
The last step of the proof is that hom(, $G) \leq$ hom(, $G)$.

If remove all the sums and simplify to just the pictures, the entire proof looks like this:



Your initial reaction to this might also be



but this way of looking at things has a lot of benefits. The main one is that it emphasizes where the action is: The proof comes down to the two ingredients of Cauchy–Schwarz and the inequality hom(, $G) \leq$ hom(, $G)$. And because of this, it makes the proof of Kruskal–Katona, which at first seems strange, into a technique that you can unleash anywhere.

Cauchy–Schwarz for graph inequalities: Take a graph F and "split" it into two parts F_1 and F_2 that glue together into F. Let F_1^2 be the graph you get when you glue two copies of F_1 together, and F_2^2 similarly. Then

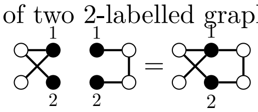
$$\mathrm{hom}(F,G)^2 \leq \mathrm{hom}(F_1^2, G)\mathrm{hom}(F_2^2, G)$$

for every graph G.

Here's a more precise way to set things up.

Definition 5. A *k-labelled graph* is a graph (which might have more than k vertices) where each of the elements of $\{1, 2, \dots, k\}$ is assigned to a distinct vertex. If F_1 and F_2 are two k-labelled graphs, the *gluing product* of F_1 and F_2 is the graph obtained by identifying vertices with the same labels in the disjoint union of F_1 and F_2.

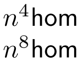
For example, here is the gluing product of two 2-labelled graphs:

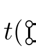
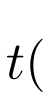
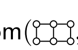
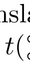


> **Activity:** C_n is the cycle graph with n vertices. Prove that
> $$\mathsf{hom}(C_6, G)^2 \le \mathsf{hom}(C_4, G)\mathsf{hom}(C_8, G)$$
> for every graph G. Use pictures. How would this proof look if you wrote it out using sums?

> **Activity:** Prove that $\mathsf{hom}($$, G)^2 \le n\,\mathsf{hom}($$, G)$.

PROBLEMS

1. Here is a different proof of Mantel's theorem which uses Cauchy–Schwarz. Suppose that G is a graph with n vertices and m edges, and x, y are vertices in G that are connected by an edge.
   (a) If you didn't yesterday, prove that $\deg(x) + \deg(y) \le n$.
   (b) Prove that
   $$\sum_{xy \in E(G)} \big(\deg(x) + \deg(y)\big) \le nm.$$
   (c) Prove that
   $$\sum_{x \in V(G)} \deg(x) = 2m \qquad \text{and} \qquad \sum_{x \in V(G)} \deg(x)^2 \le nm.$$
   (d) Use Cauchy–Schwarz to prove that
   $$\Big(\sum_{x \in V(G)} \deg(x)\Big)^2 \le n^2 m,$$
   and use this to prove Mantel's theorem.
2. Convert the previous proof into a proof by picture.
3. (a) Prove that $\mathsf{hom}($$, G)^4 \le n^4 \mathsf{hom}($$, G)$.
   (b) Prove that $\mathsf{hom}($$, G)^7 \le n^8 \mathsf{hom}($$, G)$.
   (c) Let Q_3 be the "cube graph" with 8 vertices and 12 edges. Show that $\mathsf{hom}($$, G)^{12} \le n^{16}\,\mathsf{hom}(Q_3, G)$.

Another way to set things up is to use *homomorphism density*, which is the *fraction* of maps $V(F) \to V(G)$ that are valid homomorphisms. If G has n vertices and F and k vertices, the *homomorphism density* of F in G is
$$t(F, G) := \frac{\mathsf{hom}(F, G)}{n^k}.$$
Translated to homomorphism densities, the problem 3 says that $t($$, G)^4 \le t($$, G)$ and $t($$, G)^7 \le t\mathsf{hom}($$, G)$ and $t($$, G)^{12} \le t(Q_3, G)$.

A graph F with e edges is called *Sidorenko* if $\mathsf{hom}(F, G) \ge t($$, G)^{\mathsf{e}}$ for every graph G. So every graph in problem 3 is Sidorenko. However, some graphs are not Sidorenko:

4. (a) Prove that  is not Sidorenko.
   [Hint: Find a particular graph G for which the Sidorenko inequality fails.]
   (b) Prove that every Sidorenko graph is bipartite.
   [Hint: Suppose that F is *not* bipartite. Why can't F be Sidorenko?]